

Transverse limits on the uni-directional pulse propagation approximation

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I calculate the limitations on the widely-used forward-only (uni-directional) propagation assumption by considering the effects of transverse effects (e.g. diffraction). The starting point is the scalar second order wave equation, and simple predictions are made which aim to clarify the forward-backward coupling limits on diffraction strength. The result is unsurprising, being based on the ratio of transverse and total wave vectors, but the intent is to present a derivation directly comparable to a recently published *nonlinearity* constrained limits on the uni-directional approximation [1].

I. INTRODUCTION

Most approaches to optical pulse propagation rely on an approximation where the fields only propagate forwards. Even the recently derived extensions of typical propagation methods used in nonlinear optics (e.g. [2, 3]) assume a complete decoupling between oppositely propagating fields to optimize the calculation. Moreover, those based directly on Maxwell's equations (e.g. [4, 5, 6]) or the second order wave equation (e.g. [7, 8, 9]), are often simplified to work in the forward-only limit, where backward propagating fields are set to zero. This is despite directional decompositions of Maxwell's equations (e.g. [6, 10]) indicating that *any* effect not allowed for by that decomposition couples the forward and backward waves together – and even creates a backward field if one is not present. Usually we assume that a forward wave will not generate a significant backward wave via the diffraction because we are in the paraxial limit, and even then any generated backward component is very poorly phase matched¹.

I compare predictions for propagation wave vector from uni- and bi-directional theories. Such a comparison has been done for the more important case of nonlinearity-induced forward-backward coupling [1], and used analytical expressions for carrier shocking [11, 12] in concert with simulations to examine the effects of a uni-directional approximation. Here my intention is simply to provide a complementary calculation to clarify the effects of forward-backward coupling induced by transverse (diffraction) effects. The result, while unremarkable, does clarify the bounds on the validity of propagation models using a uni-directional approximation.

Since linear dispersion and finite nonlinear response times will typically diminish any generation of a backward wave, it is clear that any model which can be assumed uni-directional on the basis of this paper will be

more so in practice. These results do not tell us whether the uni-directional approximation would be more or less robust for situations requiring vector fields (e.g. [13]), or for nonlinear effects such as self-focusing [14], or nonlinear diffraction [15] but they at least establish a point of reference.

II. BASIC THEORY

Most optical pulse problems consider a uniform and source free dielectric medium. In such cases a good starting point is the second order wave equation, which results from the substitution of the $\nabla \times \vec{H}$ Maxwell's equation into the $\nabla \times \vec{E}$ one in the source-free case (see e.g. [16]). Further, assuming linearly polarized pulses, we can use a scalar form. Defining $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ and $\partial_a \equiv \partial/\partial a$, we can write the wave equation as

$$\left[\nabla^2 - \frac{1}{c^2} \partial_t^2 \right] E(t) = \frac{4\pi}{c^2} \partial_t^2 P_T(E(t), t). \quad (1)$$

Here I have suppressed the spatial coordinates for notational simplicity; in fact we have $E(t) \equiv E(t, \vec{r})$ and the total polarization $P(E(t), t) \equiv P_T(E(t, \vec{r}), t, \vec{r})$; also $\vec{r} = (x, y, z)$. Here we will consider only isotropic linear media, which enables us to replace P_T with a refractive index; more complicated polarization behaviour is covered elsewhere [17, 18]. Thus, in the frequency domain, we can write

$$\left[\nabla^2 - \frac{n^2(\omega)\omega^2}{c^2} \right] E(t) = 0. \quad (2)$$

However, in most descriptions of pulse propagation we will want to chose a specific propagation direction (e.g. along the z -axis), and then denote the orthogonal components (i.e. along x and y) as transverse behaviour. Thus, splitting the ∇^2 operator into its propagation (z) and transverse (x, y) parts, we can rewrite the wave equation similarly:

$$[\partial_z^2 + K^2] E(\omega) = -\nabla_\perp^2 E(\omega) \quad (3)$$

$$[\partial_z^2 + K^2] E(\omega) = +k_\perp^2 E(\omega), \quad (4)$$

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¹ If the forward field has a wave vector k_0 evolving as $\exp(+ik_0z)$, the generated backward component will evolve as $\exp(-ik_0z)$. This gives a very rapid relative oscillation $\exp(-2ik_0z)$, which will quickly average to zero.

where the total wavevector is given by $K^2 = n^2\omega^2/c^2$; the transverse component is k_\perp . If we want to describe diffraction, then we can give the field some suitable beam profile $E(x, y)$, an even simpler case is that of off-axis one dimensional propagation, which requires merely a fixed value of k_\perp .

I now factorize the wave equation, a process which, while used in optics for some time [7] has only recently been used to its full potential [1, 8, 9]. Factorization takes its name from the fact that the LHS of eqn.(3) is a simple difference of squares which might be factorized, indeed this is what was done in a somewhat ad hoc fashion by Blow and Wood in 1989 [7]. Since the factors are just $\partial_z \mp \imath K$, we can see that each (by itself) would generate a forward directed wave equation, and the other a backward one. Without going into detail (although see the appendix), a rigorous factorization procedure [8, 17] allows us to define a pair of counter-propagating Greens functions, and so divide the second order wave equation into a pair of coupled *counter-propagating* first order ones.

Counter-propagating wave equations suggest counter propagating fields, so I split the electric field up accordingly into forward (E^+) and backward (E^-) parts, with $E = E^+ + E^-$. The coupled first order wave equations are

$$\partial_z E^\pm = \pm \imath K E^\pm \mp \frac{\imath k_\perp^2}{2K} [E^+ + E^-]. \quad (5)$$

The RHS now falls into two parts, which I term the underlying and residual parts [19]. First, there is the $\imath K E^\pm$ term that, by itself, would describe plane-wave like propagation. Second, the remaining part (here proportional to k_\perp^2) which can be called “residual” terms. These residual contributions, here containing the transverse effects, account for the discrepancy between the true propagation and the underlying propagation. Although here the residual component will be only a weak perturbation in e.g. the paraxial limit, the theory presented here is valid for *any* strength. Although my preference would be to use a directional fields approach [6] rather than the factorization one used here, it is difficult to describe transverse effects satisfactorily.

Note that the work of Weston examines this kind of wave-splitting with more mathematical rigour (see e.g. [20]), although without consideration of residual terms, and (at least initially) in the context of reflections and scattering. This theory was based on that from the earlier work of Beezley and Krueger [21] who applied wave-splitting concepts to optics.

A. Bi-directional (exact) case

The scalar second order wave equation given above in eqn.(1) trivially provides a total wavevector for any given direction of propagation. This is simply a sum of squares of the parallel and transverse contributions k_\parallel and k_\perp ,

so that

$$K^2 = k_\parallel^2 + k_\perp^2. \quad (6)$$

Note the reversed signs between the RHS terms of eqn.(5), the transverse part retards the propagation given by the underlying part; the net forward-direction wavevector for some k_\perp is therefore less than the total wave vector – just as would be expected.

B. Uni-directional approximation

Now I make the uni-directional assumption and set $E^- = 0$ in eqn.(5), so that we get a wave equation with underlying ($\propto K$) and residual ($\propto k_\perp^2/2K$) components, i.e.

$$\partial_z E^\pm = \pm \imath K E^\pm \mp \frac{\imath k_\perp^2}{2K} E^\pm. \quad (7)$$

Note that the diffraction term here is identical to that obtained by applying the standard paraxial approximation to propagation in a linear dispersive medium².

Alternatively, since there is no E^- field to complicate matters, I might rewrite this wave equation using a new uni-directional wave vector K_u to define *only* an underlying propagation, i.e.

$$\partial_z E^\pm = \pm \imath K_u E^\pm, \quad (8)$$

where K_u is

$$K_u = K - \frac{k_\perp^2}{2K} = K \left[1 - \frac{1}{2} \frac{k_\perp^2}{K^2} \right]. \quad (9)$$

As expected, K_u is not equivalent to the true wave vector K ; indeed we expect it to be an approximation to k_\parallel , which specifies on-axis spatial variation of the exact propagation.

III. FORWARD-BACKWARD COUPLING

In the above, we saw that the bi-directional and uni-directional models gave different propagation wave vectors. However, note that when $k_\perp^2/K^2 \ll 1$, terms of order $(k_\perp/K)^4$ or higher are negligible. We can rearrange and then approximate eqn.(6) in that limit so that

$$k_\parallel^2 = K^2 - k_\perp^2 \quad (10)$$

$$k_\parallel = K \left[1 - \frac{k_\perp^2}{K^2} \right]^{1/2} \quad (11)$$

$$\simeq K \left[1 - \frac{1}{2} \frac{k_\perp^2}{K^2} \right] = K_u. \quad (12)$$

² I. M. Besieris, private communication

Essentially what the condition $k_{\perp}^2/K^2 \ll 1$ means is that propagation effects transverse to the chosen propagation direction must occur on a scale much larger than one wavelength, or else a uni-directional approximation will fail.

It is important to note that the existence of significant forward-backward coupling does not always demand the presence or generation of a freely propagating backward wave. It is possible for the backward wave (i.e. E^-) to be dragged along by the forward one, as seen for several example in the directional fields formalism of Kinsler et al. [6]. Nevertheless, although it such a situation might give an answer correct to within a suitable scaling, in such a case the uni-directional approximation is not strictly valid.

IV. CONCLUSION

I have demonstrated one of the fundamental limits on the widely used uni-directional propagation approxima-

tion. This was done by a simple comparison of wave vectors obtained from electromagnetic scalar wave equations allowing for all three spatial dimensions; using both an exact (and hence bi-directional) model, and an approximate uni-directional model. These results are done in the same style as, and are intended to complement existing limits placed on nonlinear effects [1]; they are not intended to startle the reader with their novelty.

I have shown that the condition $k_{\perp}/K^2 \ll 1$ must hold for the uni-directional approximation to be true; even when no backward field is initially present. Unsurprisingly this is comparable to the condition for the widely used “paraxial” limit. Note, however, that the use of the paraxial approximation is rarely accompanied by a discussion of potential generation of backward propagating waves.

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Appendix: Factorizing

Here is a quick derivation of the factorization process; the z -derivative has been converted to $\imath k$, $\beta^2 = n^2\omega^2/c^2$, and the unspecified residual term is denoted Q .

$$[-k^2 + \beta^2] E = -Q \quad (13)$$

$$E = \frac{1}{k^2 - \beta^2} Q = \frac{1}{(k - \beta)(k + \beta)} \quad (14)$$

$$= \frac{-1}{2\beta} \left[\frac{1}{k + \beta} - \frac{1}{k - \beta} \right] Q. \quad (15)$$

Now $(k - \beta)^{-1}$ is a forward-like propagator for the field, and $(k + \beta)^{-1}$ a backward-like propagator. Hence write $E = E^+ + E^-$, and split the two sides up

$$E^+ + E^- = \frac{-1}{2\beta} \left[\frac{1}{k + \beta} - \frac{1}{k - \beta} \right] Q \quad (16)$$

$$E^{\pm} = \frac{\pm 1}{2\beta} \frac{1}{k \mp \beta} Q \quad (17)$$

$$[k \mp \beta] E^{\pm} = \pm \frac{1}{2\beta} \frac{1}{k \mp \beta} Q \quad (18)$$

$$\imath k E^{\pm} = \pm \imath \beta E^{\pm} \pm \frac{\imath}{2\beta} Q, \quad (19)$$

and reverting to z derivatives gives us the final form

$$\partial_z E^{\pm} = \pm \imath \beta E^{\pm} \pm \frac{\imath}{2\beta} Q. \quad (20)$$